Calculus II - Day 2

Prof. Chris Coscia, Fall 2024
Notes by Daniel Siegel

9 September 2024

1 Lecture Goals

- Use the squeeze theorem.
- Define what it means to be monotonic and state the monotonic convergence theorem.
- Define what it means for a sequence $\{a_n\}$ to "grow faster" than another sequence $\{b_n\}$ (denoted as $\{a_n\} \gg \{b_n\}$).

2 Reminders

- Gradescope HW 0: due Tuesday by midnight
- myLab HW 1: due Wednesday by noon

3 Warm-up

Find the limits of the following sequence:

a) $\lim_{n\to\infty} 2n \sin\left(\frac{1}{n}\right)$

$$
\begin{aligned}
&= \infty \cdot 0 \\
&= \lim_{n\to\infty} \frac{2\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{0}{0} \\
&\overset{\text{L'Hopital's}}{=} \lim_{n\to\infty} \frac{2\cdot\left(-\frac{1}{n^2}\right)\cos\left(\frac{1}{n}\right)}{-\frac{1}{n^2}} \\
&= \lim_{n\to\infty} 2\cos\left(\frac{1}{n}\right) \\
&= 2
\end{aligned}
$$

b) $\lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{n}} = 0^0$

Let $L = \lim_{n\to\infty}\left(\frac{1}{n}\right)^{\frac{1}{n}}$

$$\ln(L) = \lim_{n\to\infty} \ln\left[\left(\frac{1}{n}\right)^{\frac{1}{n}}\right] = \lim_{n\to\infty} \frac{1}{n}\ln\left(\frac{1}{n}\right)$$

$$= \lim_{n\to\infty} \frac{\ln\left(\frac{1}{n}\right)}{n} = \frac{-\infty}{\infty}$$

$$\overset{\text{L'Hopital's}}{=} \lim_{n\to\infty} \frac{-\frac{1}{n^2}\cdot\frac{1}{\left(\frac{1}{n}\right)}}{1}$$

$$= \lim_{n\to\infty} \frac{-1}{n} = 0$$

$$\ln(L) = 0 \quad \Rightarrow \quad L = e^0 = \boxed{1}$$

4 Limit Laws

Assume $\lim\limits_{n\to\infty} a_n = A$ and $\lim\limits_{n\to\infty} b_n = B$. Then:

1) $\lim\limits_{n\to\infty} (a_n \pm b_n) = A \pm B$
2) $\lim\limits_{n\to\infty} ca_n = c\lim\limits_{n\to\infty} a_n = c \cdot A$
3) $\lim\limits_{n\to\infty} a_n b_n = A \cdot B$
4) $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ **as long as** $B \neq 0$

Squeeze Theorem for Limits: If $a_n \leq c_n \leq b_n$ for all n and

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n,$$

then

$$\lim_{n\to\infty} c_n = L, \text{ as well.}$$

Example: Compute $\lim\limits_{n\to\infty} \frac{\cos(n)}{n^2+1}$.

Since we know that $\cos(n)$ is always between -1 and 1, and that (n^2+1) trends towards infinity, the limit will approach 0 as the denominator grows without bound.

$$\lim_{n\to\infty} \frac{\cos(n)}{n^2+1} = 0$$

Using the squeeze theorem, we compare the sequences $\{a_n\} = \left\{-\frac{1}{n^2+1}\right\}$ and $\{b_n\} = \left\{\frac{1}{n^2+1}\right\}$. For all n:

$$-\frac{1}{n^2+1} \le \frac{\cos(n)}{n^2+1} \le \frac{1}{n^2+1}$$

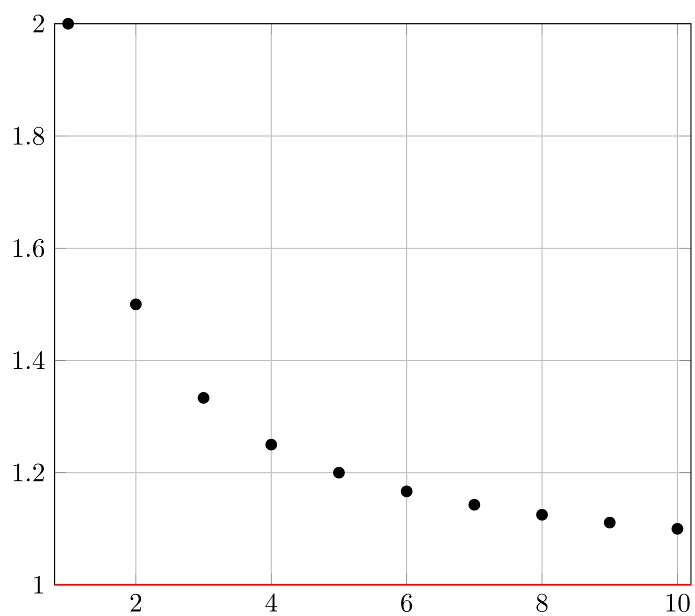
$$\begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 \end{array}$$

Squeeze Terminology

- $\{a_n\}$ is increasing if $a_{n+1} > a_n$ for all n.
- $\{a_n\}$ is nondecreasing if $a_{n+1} \ge a_n$ for all n.
- $\{a_n\}$ is decreasing if $a_{n+1} < a_n$ for all n.
- $\{a_n\}$ is nonincreasing if $a_{n+1} \le a_n$ for all n.
- $\{a_n\}$ is monotonic if it is either nonincreasing or nondecreasing.

Example: The sequence $\{1, 1, 2, 2, 3, 3, 4, 4, \ldots\}$ is not increasing (there are "ties"), but it is nondecreasing and therefore monotonic.

Example: The sequence $\left\{1 + \frac{1}{n}\right\}_{n=1}^{\infty}$ starts as $\left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\}$.
The sequence is decreasing, and therefore nonincreasing and monotonic.



Definition: $\{a_n\}$ is <u>bounded above</u> if there is a number M such that

$$a_n \leq M \text{ for every } n$$

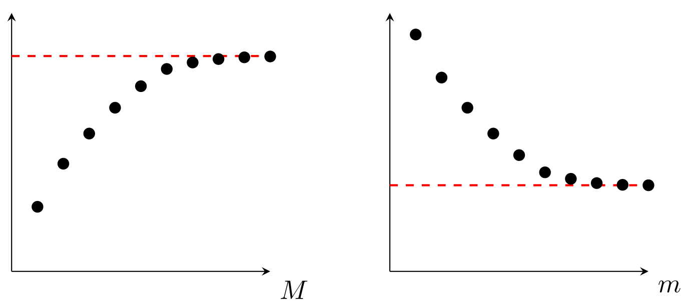
- <u>bounded below</u> if there is a number m such that

$$a_n \geq m \text{ for every } n$$

- <u>bounded</u> if it is both bounded above and bounded below.

Monotone Convergence Theorem:
Every bounded monotonic sequence converges. In fact:

1) Every nonincreasing sequence bounded above converges.



2) Every nonincreasing sequence bounded below converges.

Q: Which sequence grows faster, $\{n^2\}$ or $\{2^n\}$?
Let's look at:

$$\lim_{n\to\infty} \frac{n^2}{2^n} = \frac{\infty}{\infty}$$

$$\overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{2n}{\ln(2)\cdot 2^n} = \frac{\infty}{\infty}$$

$$\overset{\text{L'H}}{=} \lim_{n\to\infty} \frac{2}{\ln(2)^2\cdot 2^n} = 0$$

The denominator "overpowers" the numerator, making the limit 0. We write $\{2^n\} \gg \{n^2\}$ (the $\gg$ signifies that 2^n is approaching infinity faster, letting us determine which grows faster).

We say that $\{a_n\}$ "grows faster" than $\{b_n\}$ if:

$$\lim_{n\to\infty} a_n = \infty = \lim_{n\to\infty} b_n,$$

$$\textbf{and} \tag{1}$$

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0$$

OR, equivalently:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$$

We say that $\{a_n\}$ and $\{b_n\}$ have the same growth rate if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L \quad \text{for} \quad 0 < L < \infty$$

Example: Which grows faster, $\{2^n\}$ or $\{n!\}$?
Reminder: $n! = n(n-1)(n-2)\ldots 3 \cdot 2 \cdot 1$
Consider the sequence $\{\frac{2^n}{n!}\}$.

$$\frac{2^n}{n!} = \frac{2 \cdot 2 \cdot 2 \cdots 2 \cdot 2}{n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1} = \left(\frac{2}{n}\right) \cdot \left(\frac{2}{n-1}\right) \cdot \left(\frac{2}{n-2}\right) \cdots \frac{2}{2} \cdot \frac{2}{1}$$

Now, consider the next term in the sequence:

$$\frac{2^{n+1}}{(n+1)!} = \frac{2 \cdot 2^n}{(n+1) \cdot n!} = \frac{2}{n+1} \cdot \frac{2^n}{n!}$$

$$\stackrel{\sim}{\Rightarrow} \frac{2^n}{n!} \to 0 \quad \text{so} \quad \{n!\} \gg \{2^n\}$$

Example: Which grows faster: $\{n!\}$ or $\{n^n\}$?

$$n^n \gg n!$$

Note: Consider the sequence $\{\frac{n!}{n^n}\}$:

$$\frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot (n-2) \cdots 1}{n \cdot n \cdot n \cdots n} = \left(\frac{n}{n}\right) \cdot \left(\frac{n-1}{n}\right) \cdot \left(\frac{n-2}{n}\right) \cdots \left(\frac{1}{n}\right)$$

As $n \to \infty$, this product tends to 0, so we conclude that $\{n^n\} \gg \{n!\}$, meaning n^n grows faster than $n!$.

Growth rate hierarchy:

$$\{\ln(n)^q\} \ll \{n^p\} \ll \{n^p \ln(n)^r\} \ll \{n^{p+s}\} \ll \{b^n\} \ll \{n!\} \ll \{n^n\}$$